



TITLE:

ADIABATIC LIMITS OF η -INVARIANTS AND THE MEYER FUNCTION FOR SMOOTH THETA DIVISORS(Algebraic Geometry and Topology)

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CITATION:

Iida, Shuichi. ADIABATIC LIMITS OF η -INVARIANTS AND THE MEYER FUNCTION FOR SMOOTH THETA DIVISORS(Algebraic Geometry and Topology). 数理解析研究所講究録 2006, 1490: 109-126

ISSUE DATE:

2006-05

URL:

<http://hdl.handle.net/2433/58242>

RIGHT:

ADIABATIC LIMITS OF η -INVARIANTS AND THE MEYER FUNCTION FOR SMOOTH THETA DIVISORS

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1. Introduction

Let Σ_g be a closed oriented surface of genus g and let \mathcal{M}_g be the mapping class group of genus g , namely the group of all isotopy classes of orientation-preserving diffeomorphisms of Σ_g . Meyer introduced a cocycle $\tau_g : \mathcal{M}_g \times \mathcal{M}_g \rightarrow \mathbb{Z}$, called the signature cocycle or the Meyer cocycle, and he gave a signature formula for the signature of surface bundles over surfaces ([21]). Let $[\tau_g] \in H^2(\mathcal{M}_g, \mathbb{Z})$ denotes the cohomology class of τ_g . When $g = 1$, since $\mathcal{M}_1 = SL_2(\mathbb{Z})$, $H^1(SL_2(\mathbb{Z}), \mathbb{Z}) = 0$ and $3[\tau_1] = 0$, there exists a unique 1-cocycle $\phi_1 : SL_2\mathbb{Z} \rightarrow \frac{1}{3}\mathbb{Z}$ such that cobounds τ_1 . The function ϕ_1 is called the Meyer function of genus one, which has the following property: Let $\pi : Z \rightarrow X$ be a Σ_1 -bundle over a compact oriented surface with boundary $\partial Z = c_1 \amalg \cdots \amalg c_k$. Let A_1, \dots, A_k be the monodromies around each component of the boundary. Since the Picard-Lefschetz transformation along c_i is an automorphism of $H^1(\Sigma_1, \mathbb{Z})$ preserving the intersection form, one has $A_i \in SL_2(\mathbb{Z})$ by fixing a symplectic basis of $H^1(\Sigma_1, \mathbb{Z})$. Then the signature of Z , which is defined as the signature of the cup-product pairing on $H^2(Z, \partial Z, \mathbb{R})$, satisfies

$$(1) \quad \text{Sign}(Z) = \sum_{i=1}^k \phi_1(A_i).$$

The explicit formula of ϕ_1 was obtained by Meyer ([21]).

When $g = 2$, since $5[\tau_2] = 0 \in H^2(\mathcal{M}_2, \mathbb{Z}) \cong \mathbb{Z}/10\mathbb{Z}$ and $H^1(\mathcal{M}_2, \mathbb{Z}) = 0$, there exists a unique 1-cocycle $\phi_2 : \mathcal{M}_2 \rightarrow \frac{1}{5}\mathbb{Z}$ satisfying (1), for every Σ_2 -bundles over compact oriented surfaces. The function ϕ_2 is called the Meyer function of genus two.

In [1], Atiyah investigated the Meyer function ϕ_1 from the several view points. For an odd dimensional closed oriented Riemannian manifold M , let $\eta(M)$ be the η -invariant of M with respect to the signature operator of M [2]. For $\sigma \in SL_2\mathbb{Z}$, let $\pi : M_\sigma \rightarrow S^1$ be the mapping

torus associated with σ , i.e., Σ_1 -bundle over S^1 with monodromy σ . Then Atiyah showed the following identity, when M_σ is equipped with a certain metric:

$$\phi_1(\sigma) = \eta(M_\sigma)$$

Moreover, he gave several interpretation of ϕ_1 in terms of the following quantities: (1) Hirzebruch's signature defect; (2) the transformation laws of the logarithm of the Dedekind η -function; (3) the logarithm of the monodromy of Quillen's line bundle; (4) the special value of the Shimizu L -function at the origin.

In this note, we study an extension of the result of Atiyah to the case $g = 2$ and higher dimensional manifold. We shall construct a higher dimensional analogue of the Meyer function for smooth theta divisors of odd dimension.

Notation : For a complex manifold M , $T^{1,0}M$ (resp. $T^{0,1}M$) denotes the holomorphic (resp. anti-holomorphic) tangent bundle and TM denotes the real tangent bundle. We set $d^c := \frac{1}{4\pi\sqrt{-1}}(\partial - \bar{\partial})$. Hence $dd^c = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}$.

Acknowledgement : The author would like to thank Professor Nariya Kawazumi for the proof of Lemma 7.2 and Professor Tomohide Terasoma for the proof of Lemma 8.2. Special thanks are due to Professor Ken-ichi Yoshikawa for various comments and suggestions and the interests in my studies.

2. Preliminaries from Riemannian geometry

In this section, we recall some results of Riemannian geometry which will be used in the proof of the main theorem. Following [10], we define connections of fiber bundles and the connection of relative tangent bundles. Let M be a manifold and let $\pi : Z \rightarrow B$ be a fiber bundle with typical fiber M .

The *relative tangent bundle* $T(Z/B)$ is the subbundle of TZ defined by

$$T(Z/B) := \text{Ker}\{\pi_* : TZ \rightarrow \pi^*TB\}.$$

A vector of $T(Z/B)$ is said to be *vertical*.

Definition 2.1. A subbundle $T_H Z \subset TZ$ with $TZ = T(Z/B) \oplus T_H Z$ is called a *connection* of the fiber bundle $\pi : Z \rightarrow B$.

For a connection, one has $T_H Z \cong \pi^*TB$ via the projection $\pi_* : TZ \rightarrow \pi^*TB$. A vector of $T_H Z$ is said to be *horizontal*.

When Z is trivial, i.e., $Z = M \times B$, TZ is naturally isomorphic to the direct sum $(\text{pr}_1)^*TM \oplus (\text{pr}_2)^*TB$. This connection is called the *trivial connection* of the trivial fiber bundle.

Given a connection, one can define the projection $P_Z : TZ \rightarrow T(Z/B)$ with kernel $T_H Z$. We often identify P_Z with the corresponding connection $T_H Z := \text{Ker}(P_Z)$. In the rest of Section 2, we fix a connection $T_H Z$, or equivalently P_Z . One can define the pull-back of a connection, as follows: Let B' be a manifold and let $h : B' \rightarrow B$ be a C^∞ map. The fiber product $Z' := Z \times_B B' = \{(x, b) \in Z \times B' \mid \pi(x) = h(b)\}$ satisfies the following commutative diagram:

$$\begin{array}{ccc} Z' & \xrightarrow{\tilde{h}} & Z \\ \pi' \downarrow & & \downarrow \pi \\ B' & \xrightarrow{h} & B \end{array} \quad \tilde{h} = \text{pr}_1, \pi' = \text{pr}_2.$$

Since the map $P_Z \circ \tilde{h}_* : TZ' \rightarrow h^*T(Z/B)$ is surjective, $\text{Ker}(P_Z \circ \tilde{h}_*)$ is a subbundle of TZ' . Since $T(Z'/B')$ is canonically isomorphic to $h^*T(Z/B)$, the map $P_Z \circ \tilde{h}_*$ is identified with a projection from TZ' to $T(Z'/B')$.

Definition 2.2. The connection of $\pi' : Z' \rightarrow B'$ induced from $T_H Z$ by h is defined by

$$T_H Z' := \text{Ker}(P_Z \circ \tilde{h}_* : T Z' \rightarrow T(Z/B)),$$

under the identification between $T(Z'/B')$ and $h^* T(Z/B)$. The projection corresponding to $T_H Z'$ is denoted by $h^* P_Z$.

We fix a metric $g^{Z/B}$ on the relative tangent bundle, a Riemannian metric g^B on B , and the connection $T_H Z$ and the corresponding projection P_Z . We define the Riemannian metric g^Z on the total space Z by

$$g^Z := g^{Z/B} \oplus \pi^* g^B,$$

under the isomorphism $TZ \cong T(Z/B) \oplus T_H Z \cong T(Z/B) \oplus \pi^* TB$. Let ∇^Z be the Levi-Civita connection of (Z, g^Z) . We define the connection $\nabla^{Z/B}$ on $T(Z/B)$ by

$$\nabla^{Z/B} := P_Z \circ \nabla^Z.$$

Then $\nabla^{Z/B}$ preserves the metric $g^{Z/B}$.

Lemma 2.3. The connection $\nabla^{Z/B}$ is independent of a choice of g^B

Proof. See [10, Proposition 10.2] □

Lemma 2.4. Let B' be a manifold and let $h : B' \rightarrow B$ be a C^∞ -map, and set $Z' := Z \times_B B'$. Let $g^{Z'/B'} = h^* g^{Z/B}$ be the metric on $T(Z'/B')$ induced from $g^{Z/B}$, and let $P_{Z'} = h^* P_Z$ be the connection of Z' induced from P_Z . Then $\nabla^{Z'/B'} = h^* \nabla^{Z/B}$.

Proof. See [15] □

With respect to the decomposition $TZ = T(Z/B) \oplus T_H Z$, We put for $\varepsilon \in \mathbb{R}^+$

$$g^{Z,\varepsilon} := g^{Z/B} \oplus \varepsilon^{-1} \pi^* g^B.$$

The Levi-Civita connections of $(Z, g^{Z,\varepsilon})$ and (B, g^B) are denoted by $\nabla^{Z,\varepsilon}$ and ∇^B , respectively. Let $R^{Z,\varepsilon}$ and R^B be the curvature of $\nabla^{Z,\varepsilon}$ and ∇^B , respectively. Then $g^Z := g^{Z,1}$ and $\nabla^Z := \nabla^{Z,1}$. We define another connection ∇ on Z by

$$\nabla := \nabla^{Z/B} \oplus \pi^* \nabla^B,$$

and we put

$$S^{(\varepsilon)} := \nabla^{Z,\varepsilon} - \nabla \in \mathcal{A}^1(\text{End}(TZ)), \quad S := S^{(1)}.$$

Then ∇ preserves the Riemannian metric $g^{Z,\varepsilon}$, and P_Z is parallel with respect to ∇ , i.e. $\nabla \circ P_Z - P_Z \circ \nabla = 0$.

Let $\{e_1, \dots, e_k\}$ be a local orthogonal framing for $(T(Z/B), g^{Z/B})$, and let $\{f_1, \dots, f_l\}$ be a local orthogonal framing for $(T_H Z, \pi^* g^B)$.

Proposition 2.5. With respect to the splitting $TZ = T(Z/B) \oplus T_H B$, the following identity holds:

$$\lim_{\varepsilon \rightarrow 0} R^{Z,\varepsilon} = \begin{pmatrix} R^{Z/B} & P_Z(\nabla S) \\ 0 & \pi^* R^B \end{pmatrix}.$$

Proof. See [7] (3.195). □

3. η -invariants

In this section, we recall the definition and some properties of η -invariants. Let (M, g^M) be a coled oriented Riemannian manifold of dimension $(2l - 1)$. Denote the space of C^∞ k -forms on M by $\mathcal{A}^k(M)$. Let $*$: $\mathcal{A}^k(M) \rightarrow \mathcal{A}^{2l-k-1}(M)$ be the Hodge star operation with respect to g^M . The signature operator $D : \oplus_{p \geq 0} \mathcal{A}^{2p}(M) \rightarrow \oplus_{p \geq 0} \mathcal{A}^{2p}(M)$ of M is defined by

$$D : \omega \mapsto (\sqrt{-1})^l (-1)^{p+1} (*d - d*)\omega, \quad \omega \in \mathcal{A}^{2p}(M).$$

Then D is an elliptic self-adjoint differential operator of first order acting on $\oplus_{p \geq 0} \mathcal{A}^{2p}(M)$. Let $\sigma(D)$ be the spectrum of D . The η -function of M is defined by

$$\eta(s) := \sum_{\lambda \in \sigma(D) \setminus \{0\}} \frac{\text{sign} \lambda}{\lambda^s},$$

for $s \in \mathbb{C}$ with $\text{Re}(s) \gg 0$. Then $\eta(s)$ extends meromorphically to \mathbb{C} and is holomorphic at $s = 0$ by [2], [7].

Definition 3.1. The real number $\eta(0)$ is called the η -invariant of (M, g^M) and is denoted by $\eta(M, g^M)$.

Let (X, g^X) be a $4k$ -dimensional, oriented, compact, Riemannian manifold with boundary Y . Put $g^Y := g^X|_Y$ and fix a color neighborhood $U \supset Y$ such that $U \cong Y \times [0, 1)$. Assume that $g^X|_U = g^Y \oplus dt^2$ under the above isomorphism. Let ∇^L be the Levi-Civita connection of (X, g^X) .

Theorem 3.2 (Atiyah-Patodi-Singer [2]). *The following equation holds:*

$$\text{Sign}(X) = \int_X L(TX, \nabla^L) - \eta(Y, g^Y)$$

Here L denotes the Hirzebruch L -polynomial, which is a multiplicative genus associated with the power series: $L(x) := x / \tanh(x)$.

Let X, B and M be closed oriented manifolds. Let $\pi : X \rightarrow B$ be a C^∞ -submersion, whose fibers are isomorphic to M . Assume that $\dim X = 4k$. Let $g^{X/B}$ be a metric on $T(X/B)$ and let g^B be a metric on TB . Let $T_H X \subset TX$ be a connection. We identify $T_H X$ with $\pi^* TB$ via π . With respect to the decomposition $TX = T(X/B) \oplus \pi^* TB$, we define the metric on X by $g^X := g^{X/B} \oplus \pi^* g^B$ and we consider the one parameter family of metrics on X defined by

$$g^{X, \varepsilon} := g^{X/B} \oplus \varepsilon^{-1} \pi^* g^B, \quad \varepsilon \in \mathbb{R}^+.$$

Theorem 3.3 (Bismut-Cheeger, [6]). *The limit $\lim_{\varepsilon \rightarrow 0} \eta(X, g^{X, \varepsilon})$ exists.*

The limit $\lim_{\varepsilon \rightarrow 0} \eta(X, g^{X, \varepsilon})$ is called the *adiabatic limit of the η -invariants* and is denoted by $\eta^0(X)$. By definition, $\eta^0(X, g^X)$ depends on the three data: $g^{X/B}$, g^B and $T_H X$.

4. Family of smooth theta divisors

We fix the following notation. Let \mathfrak{S}_g be the Siegel upper-half space of degree g and let Γ_g be the integral symplectic group, i.e.,

$$\begin{aligned} \mathfrak{S}_g &:= \{\tau \in M(g, \mathbb{C}) \mid {}^t \tau = \tau, \text{Im} \tau > 0\} \\ \Gamma_g &:= \{\gamma \in GL(2g, \mathbb{Z}) \mid \gamma J_g {}^t \gamma = J_g\}, \end{aligned}$$

where $J_g = \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix}$ and 1_g denotes the $g \times g$ identity matrix. Γ_g acts on \mathfrak{S}_g by

$$\gamma \cdot \tau := (A\tau + B)(C\tau + D)^{-1}, \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g, \quad \tau \in \mathfrak{S}_g.$$

For $\tau \in \mathfrak{S}_g$, write $\tau = {}^t(\tau_1, \dots, \tau_g)$ and set

$$\Lambda_\tau := \mathbb{Z}e_1 \oplus \dots \oplus \mathbb{Z}e_g \oplus \mathbb{Z}\tau_1 \oplus \dots \oplus \mathbb{Z}\tau_g \subset \mathbb{C}^g$$

where $1_g = {}^t(e_1, \dots, e_g)$ and $\tau = {}^t(\tau_1, \dots, \tau_g) \in \mathfrak{S}_g$. Define the \mathbb{Z}^{2g} -action on $\mathbb{C}^g \times \mathfrak{S}_g$ by

$$(m, n) \cdot (z, \tau) := (z + m\tau + n, \tau), \quad (z, \tau) \in \mathbb{C}^g \times \mathfrak{S}_g, \quad m, n \in \mathbb{Z}^{2g}.$$

Then

$$f : \mathbb{A}_g := (\mathbb{C}^g \times \mathfrak{S}_g) / \mathbb{Z}^{2g} \rightarrow \mathfrak{S}_g$$

is the universal family of principally polarized Abelian varieties over \mathfrak{S}_g , whose fiber at τ is $A_\tau := \mathbb{C}^g / \Lambda_\tau$. For $(a, b) \in \mathbb{R}^{2g}$, $z \in \mathbb{C}^g$ and $\tau \in \mathfrak{S}_g$ we define the theta function with characteristic by

$$\vartheta_{a,b}(z, \tau) := \sum_{n \in \mathbb{Z}^{2g}} e\left(\frac{1}{2}(n+a)\tau^t(n+a) + (n+a)^t(z+b)\right),$$

where $e(t) = \exp(2\pi\sqrt{-1}t)$. Let

$$f : \Theta_{a,b} := \{(z, \tau) \in \mathbb{A}_g \mid \vartheta_{a,b}(z, \tau) = 0\} \rightarrow \mathfrak{S}_g.$$

be the *universal family of theta divisors*. For simplicity we write ϑ for $\vartheta_{0,0}$ and set $\Theta = \Theta_{0,0}$. On \mathbb{A}_g , Γ_g acts by

$$\gamma \cdot (z, \tau) := (z(C\tau + D)^{-1}, (A\tau + B)(C\tau + D)^{-1}), \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g, \quad z \in \mathbb{C}^g, \quad \tau \in \mathfrak{S}_g.$$

For any $(m, n) \in \mathbb{R}^{2g}$, we define an automorphism $t_{m,n} : \mathbb{A}_g \rightarrow \mathbb{A}_g$ by

$$(z, \tau) := (z + m\tau + n, \tau).$$

Then $t_{(m,n)}$ has no fixed points when $(m, n) \in \mathbb{R}^{2g} \setminus \mathbb{Z}^{2g}$ and the subgroup $\mathbb{Z}^{2g} \subset \mathbb{R}^{2g}$ acts trivially on \mathbb{A}_g . For $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, we define

$$\tilde{\gamma} := t_{(m,n)} \circ \gamma \in \text{Aut}(\mathbb{A}_g), \quad (m, n) := \frac{1}{2}((C^t D)_0, (A^t B)_0).$$

Then $\tilde{\gamma}$ preserves the family $f : \Theta \rightarrow \mathfrak{S}_g$.

Proposition 4.1. *For any $\gamma_1, \gamma_2 \in \Gamma_g$,*

$$\tilde{\gamma}_1 \circ \tilde{\gamma}_2 = \widetilde{\gamma_1 \gamma_2}$$

Proof. See [15] □

We set

$$g^{\mathbb{A}_g/\mathfrak{S}_g} := dz \cdot (\text{Im}\tau)^{-1} \cdot {}^t d\bar{z}.$$

Then $g^{\mathbb{A}_g/\mathfrak{S}_g}$ is a Γ_g -invariant Hermitian metric on the relative tangent bundle $T(\mathbb{A}_g/\mathfrak{S}_g)$. The next purpose of this section is to construct a Γ_g -invariant Kähler metric on $T\mathbb{A}_g$ such that $g^{\mathbb{A}_g}|_{A_\tau} = dz \cdot (\text{Im}\tau)^{-1} \cdot {}^t d\bar{z}$ for all $\tau \in \mathfrak{S}_g$.

Put $T^{2g} := \mathbb{R}^{2g} / \mathbb{Z}^{2g}$. Define a \mathbb{Z}^{2g} -action on $\mathbb{R}^{2g} \times \mathfrak{S}_g$ by $(m, n) \cdot (x, y, \tau) := (x + m, y + n, \tau)$ for $(m, n) \in \mathbb{Z}^{2g}$, $(x, y) \in \mathbb{R}^{2g}$, $\tau \in \mathfrak{S}_g$. Then $(\mathbb{R}^{2g} \times \mathfrak{S}_g) / \mathbb{Z}^{2g}$ is the trivial T^{2g} -bundle $T^{2g} \times \mathfrak{S}_g$. We define a C^∞ -map $\tilde{\rho} : \mathbb{R}^{2g} \times \mathfrak{S}_g \rightarrow \mathbb{C}^g \times \mathfrak{S}_g$ by

$$\tilde{\rho}((x, y), \tau) := (x\tau + y, \tau), \quad x, y \in \mathbb{R}^g, \quad \tau \in \mathfrak{S}_g.$$

Since $\tilde{\rho}$ is a \mathbb{Z}^{2g} -equivariant map, $\tilde{\rho}$ induces a C^∞ -isomorphism $\rho : T^{2g} \times \mathfrak{S}_g \rightarrow \mathbb{A}_g$ as T^2 -bundles over \mathfrak{S}_g . Define a Γ_g -action on $T^{2g} \times \mathfrak{S}_g$ by

$$\gamma \cdot ((x, y), \tau) := ((x, y)\gamma^{-1}, \gamma \cdot \tau), \quad \gamma \in \Gamma_g.$$

Then for any $\gamma \in \Gamma_g$, the following diagram is commutative.

$$\begin{array}{ccc} T^{2g} \times \mathfrak{S}_g & \xrightarrow{\rho} & \mathbb{A}_g \\ \gamma \downarrow & & \downarrow \gamma \\ T^{2g} \times \mathfrak{S}_g & \xrightarrow[\rho]{} & \mathbb{A}_g \end{array}$$

Since the trivial connection on $T^{2g} \times \mathfrak{S}_g$ is Γ_g -invariant, \mathbb{A}_g has the induced Γ_g -invariant connection $T_H \mathbb{A}_g \subset T \mathbb{A}_g$ via the Γ_g -equivariant isomorphism ρ . We denote the Γ_g -equivariant projection corresponding to $T_H \mathbb{A}_g$ by P_ρ . Let $P_\rho^{\mathbb{C}} : T \mathbb{A}_g \otimes \mathbb{C} \rightarrow T(\mathbb{A}_g/\mathfrak{S}_g) \otimes \mathbb{C}$ be the complexification of P_ρ . Then $P_\rho^{\mathbb{C}}$ is also Γ_g -equivariant.

Under the projection, the horizontal lift of a $(1, 0)$ (resp. $(1, 0)$) tangent vector is a $(1, 0)$ (resp. $(1, 0)$) tangent vector. Therefore the extension $P_\rho^{\mathbb{C}} : T \mathbb{A}_g \otimes \mathbb{C} \rightarrow T(\mathbb{A}_g/\mathfrak{S}_g) \otimes \mathbb{C}$ decomposes

$$(2) \quad P_\rho^{\mathbb{C}} = P_\rho^{1,0} \oplus P_\rho^{0,1},$$

under the isomorphism $T \mathbb{A}_g \otimes \mathbb{C} = T^{1,0} \mathbb{A}_g \oplus T^{0,1} \mathbb{A}_g$ and $T(\mathbb{A}_g/\mathfrak{S}_g) \otimes \mathbb{C} = T^{1,0}(\mathbb{A}_g/\mathfrak{S}_g) \oplus T^{0,1}(\mathbb{A}_g/\mathfrak{S}_g)$. Hence P_ρ induces a Γ_g -equivariant C^∞ -isomorphism

$$(3) \quad T^{1,0} \mathbb{A}_g \cong T^{1,0}(\mathbb{A}_g/\mathfrak{S}_g) \oplus f^* T^{1,0} \mathfrak{S}_g.$$

Let $g^{\mathfrak{S}_g}$ be the Bergman metric on \mathfrak{S}_g with Kähler form

$$(4) \quad \omega_{\mathfrak{S}_g} = -2\sqrt{-1} \partial \bar{\partial} \log \det \text{Im} \tau.$$

Then $g^{\mathfrak{S}_g}$ is Γ_g -invariant. Using the Γ_g -equivariant isomorphism (3), we define the Γ_g -invariant Hermitian metric $g^{\mathbb{A}_g}$ on $T \mathbb{A}_g$ by

$$g^{\mathbb{A}_g} := g^{\mathbb{A}_g/\mathfrak{S}_g} \oplus f^* g^{\mathfrak{S}_g}.$$

Theorem 4.2. *The Hermitian metric $g^{\mathbb{A}_g}$ is Kähler.*

Proof. See [15] □

We put

$$A_k(\Gamma_g, \chi) = \{f \in \mathcal{O}(\mathfrak{S}_g) \mid f(\gamma \cdot \tau) = j(\tau, \gamma)^k \chi(\gamma) f(\tau), \gamma \in \Gamma_g\}$$

where χ is a character of Γ_g and $j(\tau, \gamma) = \det(C\tau + D)$ for $\gamma \in \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. An element of $A_k(\Gamma_g, \chi)$ is called a *Siegel modular form* of weight k with character χ . In particular, an element of $A_k(\Gamma_g, 1)$ is called a Siegel modular form. Let $\mathcal{F}_g^k := \mathfrak{S}_g \times \mathbb{C}^g$ be the trivial holomorphic line bundle over \mathfrak{S}_g with the Γ_g -action

$$\gamma \cdot (\tau, \xi) := (\gamma \cdot \tau, j(\gamma, \tau)^k \xi).$$

A Siegel modular form of weight k is regarded as a Γ_g -invariant holomorphic section of \mathcal{F}_g^k . Define the *Peterson metric* on \mathcal{F}_g^k by

$$\|\xi\|_{\mathcal{F}_g^k}^2 := (\det \text{Im} \tau)^k |\xi|^2, \quad (\tau, \xi) \in \mathcal{F}_g^k.$$

By the automorphic property of $\det \text{Im}(\gamma \cdot \tau) = |j(\tau, \gamma)|^{-2} \det \text{Im} \tau$, we see that $\|\cdot\|_{\mathcal{F}_g^k}$ is Γ_g -invariant.

Let $\mathcal{N}_g := \{\tau \in \mathfrak{S}_g \mid \text{Sing} \Theta_\tau \neq \emptyset\}$ be the Andreotti-Mayer locus, which is the locus of Abelian varieties whose theta divisors is singular. The followings are known for the locus \mathcal{N}_g .

Theorem 4.3 ([12]). \mathcal{N}_g is a divisor of \mathfrak{S}_g , consisting of two irreducible components as a divisor of the modular variety $\Gamma_g \backslash \mathfrak{S}_g$:

$$\mathcal{N}_g = \theta_{\text{null},g} + 2\mathcal{N}'_g.$$

Here $\theta_{\text{null},g}$ is the zero divisor of Igusa's modular form $\chi_g(\tau)$ which is the Siegel modular form of weight $2^{g-2}(2^g + 1)$ defined as the product of all even theta constants and $\mathcal{N}'_g = \emptyset$ for $g = 2, 3$. For a generic point $\tau \in \theta_{\text{null},g}$, $\text{Sing}(\Theta_\tau)$ consists of one ordinary double point.

Theorem 4.4 ([25]). There is a Siegel cusp form $\Delta_g(\tau)$ of weight $\frac{(g+3) \cdot g!}{2}$ with zero divisor \mathcal{N}_g . By the Proposition 4.3, this implies that there exists $J_g(\tau)$ which is a Siegel modular form of weight $\frac{(g+3) \cdot g!}{4} - 2^{g-3}(2^g + 1)$ with zero divisor \mathcal{N}'_g such that

$$\Delta_g := \chi_g(\tau) J_g(\tau)^2.$$

We put $\mathfrak{S}'_g := \mathfrak{S}_g - \mathcal{N}_g$, $\Theta'_g := \Theta|_{\mathfrak{S}'_g}$. Then $f : \Theta' \rightarrow \mathfrak{S}'_g$ is a family of smooth theta divisors. Endow $T^{1,0}(\Theta'/\mathfrak{S}'_g)$ the Hermitian metric $g^{\Theta'/\mathfrak{S}'_g} := g^{\mathfrak{A}_g/\mathfrak{S}_g}|_{\Theta'}$. Let $g^{\Theta'} := g^{\mathfrak{A}_g}|_{\Theta'}$ be the restriction of the Kähler metric $g^{\mathfrak{A}_g}$. Consider $g^{\Theta'/\mathfrak{S}'_g}$ and $g^{\Theta'}$ as Riemannian metric on $T(\Theta'/\mathfrak{S}'_g)$ and $T\Theta'$. Let

$$T_H\Theta' := (T(\Theta'/\mathfrak{S}'_g))^\perp$$

be the orthogonal complement of $T(\Theta'/\mathfrak{S}'_g)$ in $T\Theta'$, which induces a connection $P_{\Theta'} : T\Theta' \rightarrow T\Theta'/\mathfrak{S}'_g$. Hence we obtain the connection $\nabla^{\Theta'/\mathfrak{S}'_g}$ on $T(\Theta'/\mathfrak{S}'_g)$ by using $g^{\Theta'/\mathfrak{S}'_g}$ and $P_{\Theta'}$ as in Section 2.2. Let ∇^h be the holomorphic Hermitian connection on $T^{1,0}(\Theta'/\mathfrak{S}'_g)$ with respect to the Hermitian metric $g^{\Theta'/\mathfrak{S}'_g}$.

Lemma 4.5. Under the C^∞ -isomorphism $T(\Theta'/\mathfrak{S}'_g) \otimes \mathbb{C} \cong T^{1,0}(\Theta'/\mathfrak{S}'_g) \oplus T^{0,1}(\Theta'/\mathfrak{S}'_g)$, the following equality of connections holds.

$$\nabla^{\Theta'/\mathfrak{S}'_g} \otimes \mathbb{C} = \nabla^h \oplus \bar{\nabla}^h$$

Proof. Let ∇^L be the Levi-Civita connection on $T\mathfrak{A}_g$ and let ∇^H be the holomorphic Hermitian connection on $T^{1,0}\mathfrak{A}_g$. Since $g^{\mathfrak{A}_g}$ is Kähler, the following equality holds ([18])

$$\nabla^L \otimes \mathbb{C} = \nabla^H \oplus \bar{\nabla}^H$$

under the isomorphism $T\mathfrak{A}_g \otimes \mathbb{C} = T^{1,0}\mathfrak{A}_g \oplus T^{0,1}\mathfrak{A}_g$. By (2), we get

$$\begin{aligned} \nabla^{\Theta'/\mathfrak{S}'_g} \otimes \mathbb{C} &= (P_\rho \nabla^L P_\rho) \otimes \mathbb{C} \\ &= P_\rho^C (\nabla^L \otimes \mathbb{C}) P_\rho^C \\ &= P_\rho^{1,0} \nabla^H P_\rho^{1,0} \oplus P_\rho^{0,1} \bar{\nabla}^H P_\rho^{0,1}. \end{aligned}$$

Since $P_\rho^{1,0} \nabla^H P_\rho^{1,0} = \nabla^h$ (see [18] Chapter I, Section 6), we get the result. \square

Let g_{1_g} be the restriction of the Hermitian metric $|dz|^2$ on $T\mathfrak{A}_g/\mathfrak{S}_g$ to the relative tangent bundle $T\Theta'/\mathfrak{S}'_g$. Let $F(T\Theta'/\mathfrak{S}'_g, g_{1_g})$ be the corresponding Chern-Weil form for $F(x)$ and the holomorphic Hermitian connection of $(T\Theta'/\mathfrak{S}'_g, g_{1_g})$.

Proposition 4.6 ([24], Proposition 2.1). The following equality holds:

$$[F(T\Theta'/\mathfrak{S}'_g, g_{1_g})]^{(g,g)} \equiv 0.$$

In particular one has

$$[f_* F(T\Theta'/\mathfrak{S}'_g, g_{1_g})]^{(1,1)} \equiv 0.$$

Let $\|\Delta_{2g}(\tau)\|^2 := (\det \text{Im} \tau)^{\frac{(2g+3) \cdot (2g)!}{2}} |\Delta_{2g}(\tau)|^2$ denote the Peterson norm of the Siegel modular form $\Delta_{2g}(\tau)$ and let B_k be the k -th Bernoulli number, i.e.,

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty} (-1)^{k+1} B_k \frac{x^{2k}}{(2k)!}.$$

Theorem 4.7. *The followig equality holds:*

$$\begin{aligned} \left[f_* L(T(\Theta' / \mathfrak{S}'_{2g}), \nabla^{\Theta' / \mathfrak{S}'_{2g}}) \right]^{(2)} &= \frac{(-1)^g 2^{2g+1} (2^{2g+2} - 1)}{(2g+1)(g+1)} B_{g+1} dd^c \log \det \text{Im} \tau \\ &= \frac{(-1)^g 2^{2g+3} (2^{2g+2} - 1)}{(2g+3)!} B_{g+1} dd^c \log \|\Delta_{2g}(\tau)\|^2. \end{aligned}$$

By Lemma 4.5 and the fact that $(\nabla^h)^2$ is a $(1, 1)$ -form, we see that the left-hand side is equal to $\left[f_* L(T^{1,0}(\Theta' / \mathfrak{S}'_{2g}), \nabla^h) \right]^{(1,1)}$. By Proposition 4.6 we obtain

$$\left[L(T^{1,0}(\Theta' / \mathfrak{S}'_{2g}), \nabla^h) \right]^{(1,1)} = -dd^c f_* [\tilde{L}(T^{1,0}(\Theta' / \mathfrak{S}'_{2g}), g_{1g}, g^{\Theta' / \mathfrak{S}'_{2g}})]^{(2g-1, 2g-1)}.$$

Hence we deduced the proof to the computation of the Bott-Chern form and we can compute it by using the same idea in [25]. Since this is rather complicated, we omit the proof.

Remark 4.8. In Section 5, it will be crucial that $dd^c \log \|\Delta_g(\tau)\|^2$ is Γ_g -invariant and that $dd^c \log \|\Delta_g(\tau)\|^2$ is an exact form as a 2-form on $\Gamma_g \backslash \mathfrak{S}'_g$.

5. The signature cocycle for smooth theta divisors

Since Γ_g acts on \mathfrak{S}'_g properly discontinuously the space $\Gamma_g \backslash \mathfrak{S}'_g$ has naturally orbifold structure and can be regarded as the moduli space of smooth theta divisors. We shall consider the orbifold fundamental group of $\Gamma_g \backslash \mathfrak{S}'_g$ and construct a 2-cocycle of this group.

In the rest of this section we fix a generic base point $* \in \mathfrak{S}'_g$, i.e., $*$ satisfies $\{\gamma \in \Gamma_g \mid \gamma * = *\} = \{\pm 1_{2g}\}$. Let (B, b) be a topological space with a base point and let $\pi : \tilde{B} \rightarrow B$ be the universal covering. Then the fundamental group $\pi_1(B, b)$ acts on \tilde{B} as the deck transformation. Fix a lift $\tilde{b} \in \tilde{B}$ of $b \in B$. We set

$$[B, \Gamma_g \backslash \mathfrak{S}'_g]^{\text{orb}} := \{(p, \beta) \mid p : \tilde{B} \rightarrow \mathfrak{S}'_g, \beta : \pi_1(B, b) \rightarrow \Gamma_g, \text{ s.t. } p(\tilde{b}) = *, p(\gamma \cdot x) = \beta(\gamma) \cdot p(x)\} / \sim.$$

Here the relation $(p_0, \beta_0) \sim (p_1, \beta_1)$ holds if and only if $\beta_0 = \beta_1$ and there is a map $\tilde{p} : \tilde{B} \times [0, 1] \rightarrow \mathfrak{S}'_g$ such that $\tilde{p}(x, 0) = p_0$, $\tilde{p}(x, 1) = p_1$ and $\tilde{p}(\gamma \cdot x, t) = \beta(\gamma) \cdot \tilde{p}(x, t)$ for any $\gamma \in \Gamma_g$, $x \in \tilde{B}$, $t \in [0, 1]$.

Definition 5.1. We define the *orbifold fundamental group* of $\Gamma_g \backslash \mathfrak{S}'_g$ by

$$\begin{aligned} S_g &:= [S^1, \Gamma_g \backslash \mathfrak{S}'_g]^{\text{orb}} \\ &= \{(\alpha, \gamma) \mid \gamma \in \Gamma_g, \alpha : \mathbb{R} \rightarrow \mathfrak{S}'_g, \text{ s.t. } \alpha(0) = *, \alpha(t) = \gamma \cdot \alpha(t+1), t \in \mathbb{R}\} / \sim. \end{aligned}$$

Then

$$S_g = \{(\alpha, \gamma) \mid \gamma \in \Gamma_g, \alpha : [0, 1] \rightarrow \mathfrak{S}'_g, \text{ s.t. } \alpha(0) = \gamma \cdot \alpha(1) = *\} / \sim.$$

Here $(\alpha_0, \gamma_0) \sim (\alpha_1, \gamma_1)$ if and only if $\gamma_0 = \gamma_1$ and there exists a homotopy $\alpha(s, t) : [0, 1] \times [0, 1] \rightarrow \mathfrak{S}'_g$ connecting α_0 and α_1 , such that $\alpha(s, 0) = \gamma_0 \cdot \alpha(s, 1) = *$ for $s \in [0, 1]$.

The group law of S_g is defined as follows. Let $[(\alpha_1, \gamma_1)], [(\alpha_2, \gamma_2)] \in S_g$. Then $\gamma_2^{-1} \cdot \alpha_1$ is a path from $\gamma_2^{-1} \cdot *$ to $(\gamma_1 \gamma_2)^{-1} \cdot *$. We define the new path $\alpha : [0, 1] \rightarrow \mathfrak{S}'_g$ by $\alpha(t) := \alpha_2(2t)$ for $0 \leq t \leq \frac{1}{2}$, $\alpha(t) := \gamma_2^{-1} \cdot \alpha_1(2t-1)$ for $\frac{1}{2} \leq t \leq 1$. Then we define $[(\alpha_1, \gamma_1)] \cdot [(\alpha_2, \gamma_2)] := [(\alpha, \gamma_1 \gamma_2)] \in S_g$.

Let $p : S_g \rightarrow \Gamma_g$ be the projection to the second factor. Since the kernel of p is isomorphic to $\pi_1(\mathfrak{S}'_g, *)$, we have an exact sequence

$$(5) \quad 1 \rightarrow \pi_1(\mathfrak{S}'_g, *) \rightarrow S_g \rightarrow \Gamma_g \rightarrow 1.$$

Remark 5.2. When $g = 1$, $\Gamma_1 \backslash \mathfrak{S}'_1 = SL_2\mathbb{Z} \backslash \mathfrak{S}_1$ is the moduli space of curves of genus 1 and $S_1 = \mathcal{M}_1$. When $g = 2$, $\Gamma_2 \backslash \mathfrak{S}'_2$ is the moduli space of curves of genus 2 by the Torelli theorem and $S_2 = \mathcal{M}_2$.

Recall that a $\pi_1(B, b)$ -equivariant map $f : (\tilde{B}, \tilde{b}) \rightarrow (\mathfrak{S}'_g, *)$ induces the homomorphism of groups $f_* : \pi_1(B, b) \rightarrow S_g$ such that $f_*([c]) = [f \circ c]$ for $[c] \in \pi_1(B, b)$.

Proposition 5.3. *Let (B, b) be a compact oriented surface with base point and with non empty boundary. Then the map*

$$[B, \Gamma_g \backslash \mathfrak{S}'_g]^{orb} \ni [f] \mapsto f_* \in \text{Hom}(\pi_1(B, b), S_g).$$

is a bijection.

Proof. It is known that B is homotopy equivalent to an n -bouquet $\vee_{k=1}^n S_k^1$ for some n and the fundamental group $\pi_1(B, b) \cong \pi_1(\vee_{k=1}^n S_k^1, o)$ is isomorphic to the free group of rank n . Hence we get

$$[B, \Gamma_g \backslash \mathfrak{S}'_g]^{orb} \simeq [\vee_{k=1}^n S_k^1, \Gamma_g \backslash \mathfrak{S}'_g]^{orb} \simeq \text{Hom}(\pi_1(\vee_{k=1}^n S_k^1, o), S_g) \simeq \text{Hom}(\pi_1(B, b), S_g).$$

which completes the proof. \square

In the rest of this section we assume that $B = S^2 - \Pi_{k=1}^3 D_k$, where D_1, D_2, D_3 are mutually disjoint open discs. Since B is homotopy equivalent to a 2-bouquet $\pi_1(B, b)$ is the free group of rank 2. Let g_1, g_2 be generators of $\pi_1(B, b)$ represented by the loops which are mutually homotopy equivalent to $\partial D_1, \partial D_2$. By Proposition 5.3 we have a bijection

$$(6) \quad [B, \Gamma_g \backslash \mathfrak{S}'_g]^{orb} \simeq S_g \times S_g,$$

which is given by $[f] \mapsto (f_*(g_1), f_*(g_2)) \in S_g \times S_g$ for $[f] \in [B, \Gamma_g \backslash \mathfrak{S}'_g]^{orb}$.

For $[f] \in [B, \Gamma_g \backslash \mathfrak{S}'_g]^{orb}$ the fiber product $\pi : \tilde{B} \times_f \Theta \rightarrow \tilde{B}$ is a $\pi_1(B, b)$ -equivariant fiber bundle because $f : \tilde{B} \rightarrow \mathfrak{S}'_g$ is a $\pi_1(B, b)$ -equivariant map. We get the fiber bundle $\pi : (\tilde{B} \times_f \Theta) / \pi_1(B, b) \rightarrow B$, which is uniquely determined by $[f] \in [B, \Gamma_g \backslash \mathfrak{S}'_g]^{orb}$ up to an isomorphism and which is $2g$ -dimensional compact oriented manifold with boundary. For $(\sigma_1, \sigma_2) \in S_g \times S_g$, Let $\pi : X(\sigma_1, \sigma_2) \rightarrow B$ denote the corresponding fiber bundle under the isomorphism (6).

Definition 5.4. Define the map $c_{2g} : S_{2g} \times S_{2g} \rightarrow \mathbb{Z}$ by

$$c_{2g}(\sigma_1, \sigma_2) := \text{Sign}(X(\sigma_1, \sigma_2)).$$

We call c_{2g} the *signature cocycle for smooth theta divisors*.

Remark 5.5. We only consider the case of an even genus because in the case of an odd genus $\text{Sign}(X(\sigma_1, \sigma_2))$ always vanishes.

Lemma 5.6. *The following relation holds:*

$$c_{2g}(\sigma_1, \sigma_2) + c_{2g}(\sigma_1\sigma_2, \sigma_3) = c_{2g}(\sigma_2, \sigma_3) + c_{2g}(\sigma_2\sigma_3, \sigma_1),$$

for any $\sigma_1, \sigma_2, \sigma_3 \in S_{2g}$. In particular, c_{2g} is a 2-cocycle of the group S_{2g} ([11]).

Proof. By the same argument in [1], we obtain the assertion. \square

Let $[c_{2g}] \in H^2(S_{2g}, \mathbb{Z})$ be the cohomology class of c_{2g} . When $g = 1$, c_2 is the Meyer cocycle.

6. Construction of the Meyer function

Let $\sigma = [(\alpha, \gamma)]$ be an element of S_{2g} , where $\alpha : \mathbb{R} \rightarrow \mathfrak{S}'_{2g}$ and $\gamma \in \Gamma_{2g}$. Let $\mathbb{R} \times_{\alpha} \Theta'$ be the fiber product, which has a natural $\pi_1(S^1)$ -action. We define the *mapping torus* M_{σ} for σ by

$$\pi : M_{\sigma} := (\mathbb{R} \times_{\alpha} \Theta') / \pi_1(S^1) \rightarrow S^1.$$

Since the metric $g^{\Theta'/\mathfrak{S}'_{2g}}$ on $T(\Theta'/\mathfrak{S}'_{2g})$ and the connection $P_{\Theta'}$ on Θ' are Γ_{2g} -invariant and the map $p : \widetilde{S^1} = \mathbb{R} \rightarrow \mathfrak{S}'_{2g}$ is $\pi_1(S^1)$ -equivariant, the metric g^{M_{σ}/S^1} on $T(M_{\sigma}/S^1)$ and the connection P_{σ} on M_{σ} are naturally induced via the map p . Using the connection P_{σ} we define the 1-parameter family of Riemannian metrics $\{g^{M_{\sigma}, \varepsilon}\}_{\varepsilon > 0}$ on M_{σ} by

$$g^{M_{\sigma}, \varepsilon} := g^{M_{\sigma}/S^1} \oplus \varepsilon^{-1} \pi^* dt^2, \quad \varepsilon \in \mathbb{R}_{>0}.$$

Here we regard S^1 as \mathbb{R}/\mathbb{Z} and $t \in \mathbb{R}$ as a coordinate of S^1 . By the theorem 3.3, the adiabatic limit

$$\eta^0(M_{\sigma}, g^{M_{\sigma}, \varepsilon}) := \lim_{\varepsilon \rightarrow 0} \eta(M_{\sigma}, g^{M_{\sigma}, \varepsilon})$$

exists. Recall that the Siegel modular form $\Delta_{2g}(\tau)$ with zero divisors \mathcal{N}_{2g} (see Section 3.3). Since the 1-form $d^c \log \|\Delta_{2g}(\tau)\|^2$ is Γ_{2g} -invariant the pull-back $p^* d^c \log \|\Delta_{2g}(\tau)\|^2$ can be regarded as a 1-form on S^1 .

Definition 6.1. For $\sigma \in S_{2g}$ we fix (p, γ) which represents $\sigma = [(p, \gamma)]$, where $\gamma \in \Gamma_{2g}$ and $p : \mathbb{R} \rightarrow \mathfrak{S}'_{2g}$. we set

$$\Phi_{2g}(p, \gamma) := \eta^0(M_{\sigma}, g^{M_{\sigma}}) + \frac{(-1)^g 2^{2g+3} (2^{2g+2} - 1) B_{g+1}}{(2g+3)!} \int_{S^1} p^* d^c \log \|\Delta_{2g}(\tau)\|^2.$$

The following theorem is the main result of this paper.

Theorem 6.2. (a) The value $\Phi_{2g}(p, \gamma)$ is independent of a choice of (p, γ) which represents $\sigma \in S_{2g}$. In particular Φ_{2g} is a function on S_{2g} .

(b) The cocycle $-c_{2g}$ is the coboundary of the function Φ_{2g} . In particular $[c_{2g}] \otimes \mathbb{Q} = 0 \in H^2(S_{2g}, \mathbb{Z})$.

As a corollary of the Theorem 6.2, it follows that $\phi_2 = \Phi_2$ by the uniqueness of Meyer's function of genus 2. On the other hand, $\Delta_2(\tau)$ coincides with the Igusa's modular form $\chi_2(\tau)$ ([25]), which is the product of all even theta constants. Then we can derive the following formula:

Corollary 6.3 ([15]). Let $\sigma = [(p, \gamma)]$ be an element of $S_2 = \mathcal{M}_2$ as before. Then we have

$$\phi_2(\sigma) = \eta^0(M_{\sigma}, g^{M_{\sigma}, \varepsilon}) - \frac{2}{15} \int_{S^1} p^* d^c \log \|\chi_2(\tau)\|^2.$$

Proof of Theorem 6.2. (a) Assume that (p_0, γ) and (p_1, γ) represents the same element $\sigma \in S_{2g}$. Put $I := [0, 1]$. There is a map

$$\tilde{p} : I \times \mathbb{R} \rightarrow \mathfrak{S}'_{2g}$$

which satisfies $\tilde{p}(s, 0) = *$ for $s \in I$ and $\tilde{p}(s, t) = \gamma \cdot \tilde{p}(s, t+1)$ for $(s, t) \in I \times \mathbb{R}$ and the following condition

$$(7) \quad \tilde{p}(s, t) = p_0(t), \quad s \in [0, \frac{1}{3}) \quad \text{and} \quad \tilde{p}(s, t) = p_1(t), \quad s \in (\frac{2}{3}, 1].$$

Since \tilde{p} is $\pi_1(I \times \mathbb{R})$ -equivariant, the fiber product $(I \times \mathbb{R}) \times_{\tilde{p}} \Theta'$ has the $\pi_1(I \times S^1)$ -action and the quotient space

$$\bar{\pi} : \bar{M}_{\sigma} := (I \times \mathbb{R}) \times_{\tilde{p}} \Theta' / \pi_1(I \times S^1) \rightarrow I \times S^1$$

has the induced metric $g^{\bar{M}_\sigma/I \times S^1}$ on $T(\bar{M}_\sigma/I \times S^1)$ from the metric $g^{\Theta'}/\mathfrak{S}'_2$ and the connection \bar{P}_σ on \bar{M}_σ from the connection $P_{\Theta'}$ mutually via the map p . Using the connection \bar{P}_σ we set

$$g^{\bar{M}_\sigma, \varepsilon} := g^{\bar{M}_\sigma/I \times S^1} \oplus \varepsilon^{-1} \pi^*(ds^2 \oplus dt^2), \quad \varepsilon \in \mathbb{R}_{>0}.$$

Let $g_i^{M_\sigma, \varepsilon}$ be the metrics on M_{σ_i} induced from the map p_i for $i = 0, 1$ as above. The condition (7) implies that

$$g^{\bar{M}_\sigma, \varepsilon}|_{[0, \frac{1}{3}) \times S^1} = g_0^{M_\sigma, \varepsilon} \oplus \varepsilon^{-1} dt^2, \quad g^{\bar{M}_\sigma, \varepsilon}|_{(\frac{2}{3}, 1] \times S^1} = g_1^{M_\sigma, \varepsilon} \oplus \varepsilon^{-1} dt^2.$$

Then we can apply the Atiyah-Patodi-Singer's index theorem to $(\bar{M}_\sigma, g^{\bar{M}_\sigma, \varepsilon})$:

$$(8) \quad \text{Sign}(\bar{M}_\sigma) = \int_{I \times S^1} \bar{\pi}_* L(T\bar{M}_\sigma, g^{\bar{M}_\sigma, \varepsilon}) - (\eta(M_\sigma, g_0^{M_\sigma, \varepsilon}) - \eta(M_\sigma, g_1^{M_\sigma, \varepsilon})).$$

Since \bar{M}_σ is isomorphic to the product $M_\sigma \times I$, we have (see [3]),

$$(9) \quad \text{Sign}(\bar{M}_\sigma) = \text{Sign}(M_\sigma) \times \text{Sign}(I) = 0.$$

By Proposition 2.4 and the Proposition 2.5, we get

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{I \times S^1} \bar{\pi}_* L(T\bar{M}_\sigma, g^{\bar{M}_\sigma, \varepsilon}) &= \int_{I \times S^1} \bar{\pi}_* \left(L(T(\bar{M}_\sigma/(I \times S^1))) \cdot \bar{\pi}^* L(T(I \times S^1)) \right) \\ &= \int_{I \times S^1} \left(\bar{\pi}_* L(T(\bar{M}_\sigma/(I \times S^1)), \nabla^{\bar{M}_\sigma/(I \times S^1)}) \right)^{(2)} \\ &= \int_{I \times S^1} \left[\bar{\pi}_* \bar{p}^* L(T(\Theta'/\mathfrak{S}'_2), \nabla^{\Theta'/\mathfrak{S}'_2}) \right]^{(2)} \\ (10) \quad &= \int_{I \times S^1} \bar{p}^* \left[\bar{\pi}_* L(T(\Theta'/\mathfrak{S}'_2), \nabla^{\Theta'/\mathfrak{S}'_2}) \right]^{(2)} \end{aligned}$$

where $\nabla^{\bar{M}_\sigma/(S^1 \times I)}$ is the connection on the relative tangent bundle $T(\bar{M}_\sigma/(S^1 \times I))$ associated with $g^{\bar{M}_\sigma/(S^1 \times I)}$ and \bar{P}_σ and we used the commutativity of fiber integrals and base changes in the last equality. By the Proposition 4.7, we have

$$\begin{aligned} (11) \quad & \int_{I \times S^1} \bar{p}^* \left[\bar{\pi}_* L(T(\Theta'/\mathfrak{S}'_2), \nabla^{\Theta'/\mathfrak{S}'_2}) \right]^{(2)} \\ &= -\frac{2^{2g+3}(2^{2g+2}-1)B_{2g+2}}{(2g+3)!} \int_{I \times S^1} \bar{p}^* d^c \log \|\Delta_{2g}(\tau)\|^2 \\ &= -\frac{2^{2g+3}(2^{2g+2}-1)B_{2g+2}}{(2g+3)!} \left(\int_{\{1\} \times S^1} p_1^* d^c \log \|\Delta_{2g}(\tau)\|^2 - \int_{\{0\} \times S^1} p_0^* d^c \log \|\Delta_{2g}(\tau)\|^2 \right), \end{aligned}$$

where we used the Γ_{2g} -invariance of the 1-form $d^c \log \|\Delta_{2g}(\tau)\|^2$ in the last equality. By (25) ~ (12) and the Definition 6.1, we obtain

$$0 = \Phi_{2g}(p_1, \gamma) - \Phi_{2g}(p_0, \gamma),$$

which completes the proof of (a).

(b) Let $\sigma_1 = [(p_1, \gamma_1)]$, $\sigma_2 = [(p_2, \gamma_2)]$, $\sigma_3 := (\sigma_1 \sigma_2)^{-1} = (p_3, (\gamma_1 \gamma_2)^{-1}) \in S_{2g}$. Set $B := S^2 - \coprod_{k=1}^3 D_k$. Recall that the fiber bundle $\pi : X(\sigma_1, \sigma_2) \rightarrow B$ for σ_1, σ_2 defined at the Section 3.2. By the definition of Φ_{2g} , we have $\Phi_{2g}(\sigma^{-1}) = -\Phi_{2g}(\sigma)$ for any $\sigma \in S_{2g}$. Therefore to show that $-c_{2g}$ is the coboundary of Φ , we have to show that

$$(12) \quad \text{Sign}(X(\sigma_1, \sigma_2)) = -\sum_{i=1}^3 \Phi_{2g}(\sigma_i)$$

Let U_i be the neighborhood of ∂D_i in B such that $U_i \cong [0, 1) \times \partial D_i$. Let $\beta_i : \tilde{U}_i \cong [0, 1) \times \mathbb{R} \rightarrow \tilde{B}$ be the lift of the map $U_i \hookrightarrow B$. Let $g_1, g_2 \in \pi_1(B, b)$ be the generators represented by the loops $\partial D_1, \partial D_2$. Let $[(p, \alpha)] \in [B, \Gamma_{2g} \backslash \mathfrak{S}'_{2g}]^{orb}$ be the corresponding element for $(\sigma_1, \sigma_2) \in S_{2g} \times S_{2g}$ under the isomorphism (6) where $\alpha : \pi_1(B, b) \rightarrow \Gamma_{2g}$ is a group homomorphism and $p : \tilde{B} \rightarrow \mathfrak{S}'_{2g}$ is a $\pi_1(B, b)$ -equivariant homomorphism preserving the basepoint. Since $\partial D_1, \partial D_2$ and ∂D_3 are homotopy equivalent to the loops which represent g_1, g_2 and $(g_1 g_2)^{-1} \in \pi_1(B, b)$ we can assume that

$$(13) \quad p \circ \beta_i|_{\tilde{U}_i}(s_i, t) = p_i(t), \quad (s_i, t) \in \tilde{U}_i \cong [0, 1) \times \mathbb{R}, \quad i = 1 \sim 3.$$

Let $g^{X(\sigma_1, \sigma_2)/B}$ and $P_{X(\sigma_1, \sigma_2)}$ be the metric on $TX(\sigma_1, \sigma_2)$ and the connection on $X(\sigma_1, \sigma_2)$ induced from the metric $g^{\Theta'/\mathfrak{S}'_{2g}}$ and the connection $P_{\Theta'}$ via the map p . Let g^B be the metric on TB such that $g^B|_{U_i} = ds_i^2 \oplus dt^2$. Using the connection $P_{X(\sigma_1, \sigma_2)}$ we define the metric on $TX(\sigma_1, \sigma_2)$ by

$$g^{X(\sigma_1, \sigma_2), \varepsilon} := g^{X(\sigma_1, \sigma_2)/B} \oplus \varepsilon^{-1} \pi^* g^B, \quad \varepsilon \in \mathbb{R}_{>0}.$$

Let $g^{M_{\sigma_i}, \varepsilon}$ be the metric on M_{σ_i} induced from p_i for $i = 1 \sim 3$ as above. Let $\nabla^{X(\sigma_1, \sigma_2)/B}$ be the connection on $T(X(\sigma_1, \sigma_2))$ defined by the metric $g^{X(\sigma_1, \sigma_2)/B}$ and the connection $P_{X(\sigma_1, \sigma_2)}$. Since the condition (13) implies that the metric $g^{X(\sigma_1, \sigma_2), \varepsilon}$ is a product metric near the boundary of $X(\sigma_1, \sigma_2)$ we can apply the Atiyah-Patodi-Singer's index theorem to $(X(\sigma_1, \sigma_2), g^{X(\sigma_1, \sigma_2), \varepsilon})$:

$$\begin{aligned} \text{Sign}(X(\sigma_1, \sigma_2)) &= \int_{X(\sigma_1, \sigma_2)} L(TX(\sigma_1, \sigma_2), g^{X(\sigma_1, \sigma_2), \varepsilon}) - \sum_{i=1}^3 \eta(M_{\sigma_i}, g^{M_{\sigma_i}, \varepsilon}) \\ &= \int_B \pi_* L(T(X(\sigma_1, \sigma_2)/B), \nabla^{X(\sigma_1, \sigma_2)/B}) - \sum_{i=1}^3 \eta^0(M_{\sigma_i}, g^{M_{\sigma_i}, \varepsilon}) \\ &= \int_B p^* \left[f_* L(T(\Theta'/\mathfrak{S}'_{2g}), \nabla^{\Theta'/\mathfrak{S}'_{2g}}) \right]^{(2)} - \sum_{i=1}^3 \eta^0(M_{\sigma_i}, g^{M_{\sigma_i}, \varepsilon}) \\ &= - \sum_{i=1}^3 \int_{\partial D_i} \frac{2^{2g+3}(2^{2g+2} - 1)B_{2g+2}}{(2g+3)!} d^c \log \|\Delta_{2g}(\tau)\|^2 \\ &\quad - \sum_{i=1}^3 \eta^0(M_{\sigma_i}, g^{M_{\sigma_i}, \varepsilon}) \\ &= - \sum_{i=1}^3 \Phi_{2g}(\sigma_i) \end{aligned}$$

which completes the proof of (b). \square

7. The first cohomology of S_g

The uniqueness of a 1-cocycle that cobounds the 2-cocycle c_{2g} is equivalent to the vanishing of $H^1(S_{2g}, \mathbb{Z})$. In deed, if there is another 1-cocycle $\Phi'_{2g} : S_{2g} \rightarrow \mathbb{R}$ that cobounds c_{2g} , the difference $\Phi_{2g} - \Phi'_{2g}$ is an element of $\text{Hom}(S_{2g}, \mathbb{R}) \cong H^1(S_{2g}, \mathbb{R})$. While $H^1(S_1, \mathbb{Z}) = H^1(S_2, \mathbb{Z}) = 0$, the uniqueness no longer valid for higher genus.

Theorem 7.1. *The following holds:*

$$H^1(S_g, \mathbb{Z}) = \begin{cases} 0 & 1 \leq g \leq 3, \\ \mathbb{Z} & g \geq 4. \end{cases}$$

In particular, the cochain cobounding the signature cocycle c_{2g} is not unique when $g \geq 2$.

By (5) and [11], we have the 5-term exact sequence

$$(14) \quad 1 \rightarrow H^1(\Gamma_g, \mathbb{Z}) \rightarrow H^1(S_g, \mathbb{Z}) \rightarrow H^1(\pi_1(\mathfrak{S}'_g, *), \mathbb{Z})^{\Gamma_g} \xrightarrow{\delta} H^2(\Gamma_g, \mathbb{Z}) \rightarrow H^2(S_g, \mathbb{Z}).$$

We have $H^1(\Gamma_g, \mathbb{Z}) = 0$ for $g \geq 1$ and $H^2(\Gamma_g, \mathbb{Z}) = \mathbb{Z}$ for $g \geq 3$. By the Hurwitz theorem we see that

$$(15) \quad H^1(\pi_1(\mathfrak{S}'_g, *), \mathbb{Z}) \cong H^1(\mathfrak{S}'_g, \mathbb{Z}).$$

Lemma 7.2. *Let X be a connected complex manifold of $\dim_{\mathbb{C}} X \geq 2$. Assume that*

$$(16) \quad H^1(X, \mathbb{Z}) = H^2(X, \mathbb{Z}) = 0.$$

Let $D = \sum_{\lambda \in \Lambda} n_{\lambda} D_{\lambda}$ be a divisor on X such that $n_{\lambda} \neq 0$ and D_{λ} is irreducible for all $\lambda \in \Lambda$. Then

$$H^1(X - D, \mathbb{Z}) \cong \mathbb{Z}^{\Lambda}.$$

The generator of the cohomology $H^1(X - D, \mathbb{Z})$ corresponding to $\lambda \in \Lambda$ is represented by the map $l_{\lambda} \mapsto 1$ and $l_{\mu} \mapsto 0$ for $\mu \neq \lambda \in \Lambda$, where l_{μ} denotes the loop around a small disk and intersecting D_{μ} transversally.

Proof. Since the real codimension of $\text{Sing} D$ in X is greater than or equal to 4, we have $\pi_k(X, X - \text{Sing} D, *) = 0$ for $1 \leq k \leq 3$. The relative Hurwitz theorem asserts that $H_k(X, X - \text{Sing} D, \mathbb{Z}) = 0$ for $k \leq 3$. Hence $H^k(X, X - \text{Sing} D, \mathbb{Z}) = 0$ for $k \leq 3$, which together with the cohomology exact sequence for the triple $(X, X - \text{Sing} D, X - D)$, yields that

$$(17) \quad H^2(X, X - D, \mathbb{Z}) \cong H^2(X - \text{Sing} D, X - D, \mathbb{Z}).$$

By the cohomology exact sequence for the pair $(X, X - D)$ and (16), we obtain

$$(18) \quad H^1(X - D, \mathbb{Z}) \cong H^2(X, X - D, \mathbb{Z}).$$

Since $D - \text{Sing} D$ is a closed submanifold in $X - \text{Sing} D$ and $X - D = (X - \text{Sing} D) - (D - \text{Sing} D)$, the Thom isomorphism asserts that

$$(19) \quad H^2(X - \text{Sing} D, X - D, \mathbb{Z}) \cong H^0(D - \text{Sing} D, \mathbb{Z}).$$

By the irreducibility of D_{λ} , $D_{\lambda} - \text{Sing} D_{\lambda}$ is path connected so that

$$(20) \quad H^0(D - \text{Sing} D, \mathbb{Z}) \cong \mathbb{Z}^{\Lambda}.$$

The result follows from (17)~(20). □

Lemma 7.3. *The following holds:*

$$H^1(\pi_1(\mathfrak{S}'_g, *), \mathbb{Z})^{\Gamma_g} = \begin{cases} \mathbb{Z} & 1 \leq g \leq 3 \\ \mathbb{Z}^{\oplus 2} & g \geq 4. \end{cases}$$

By regarding $H^1(\mathfrak{S}'_g, \mathbb{C})$ as the de Rham cohomology group, the image of the generators under the natural map $H^1(\mathfrak{S}'_g, \mathbb{Z}) \rightarrow H^1(\mathfrak{S}'_g, \mathbb{C})$ are represented by the 1-forms $\frac{1}{2\pi\sqrt{-1}} d\log \chi_g(\tau)$ and $\frac{1}{2\pi\sqrt{-1}} d\log J_g(\tau)$. Here $J_g(\tau) \equiv 1$ and hence $d\log J_g(\tau) = 0$ for $1 \leq g \leq 3$.

Proof. By Proposition 4.3, Proposition 4.4, the isomorphism (15) and Lemma 7.2, we get the assertion. □

Recall that the automorphic factor $j(\tau, \gamma)$ is a nowhere vanishing holomorphic function on \mathfrak{S}_g . Since \mathfrak{S}_g is simply connected, the logarithm of $j(\tau, \gamma)$ makes sense. Choose a branch of the logarithm of $j(\tau, \gamma)$ and denote it by $\log_\sigma j(\tau, \gamma)$ for $\gamma \in \Gamma_g$. Define the function $\lambda_\sigma : \Gamma_g \times \Gamma_g \rightarrow \mathbb{Z}$ by

$$(21) \quad \lambda_\sigma(A, B) := \frac{1}{2\pi\sqrt{-1}} \{ \log_\sigma j(\tau, AB) - \log_\sigma j(B \cdot \tau, A) - \log_\sigma j(\tau, B) \}, \quad (A, B) \in \Gamma_g \times \Gamma_g.$$

Lemma 7.4. *The function λ_σ is a 2-cocycle of Γ_g , whose cohomology class generates $H^2(\Gamma_g, \mathbb{Z})$.*

Proof. For $g = 1$ see [4]. When $g \geq 1$, we follow [4]. Let $G := Sp(2g, \mathbb{R})$ be the symplectic group and let G^δ be the same group endowed with the discrete topology. Let $u \in H^2(G^\delta, \mathbb{Z})$ be the cohomology class corresponding to the universal covering

$$0 \rightarrow \mathbb{Z} \rightarrow \tilde{G} \rightarrow G \rightarrow 1.$$

We choose the branch $\log_\sigma j(\tau, \gamma)$ satisfying

$$(22) \quad \text{Im } \log_\sigma j(\sqrt{-1} \cdot 1_{2g}, \gamma) \in [0, 2\pi).$$

Since the function $\bar{\lambda}_\sigma$ is measurable, the cohomology class $[\bar{\lambda}_\sigma]$ is a constant multiple of u by [20]. Therefore it suffices to determine the restriction of the cohomology class $[\bar{\lambda}_\sigma]$ to the maximal compact subgroup of G . We shall identify the unitary group $U(g)$ with the maximal compact subgroup of G by the inclusion map defined as

$$\iota : U(g) \ni Z \mapsto \begin{pmatrix} \text{Re } Z & \text{Im } Z \\ -\text{Im } Z & \text{Re } Z \end{pmatrix} \in G, \quad Z \in U(g).$$

Since $j(\sqrt{-1} \cdot 1_{2g}, \iota(Z)) = \det(Z)^{-1}$ for $Z \in U(g)$ and the isotropy subgroup at $\sqrt{-1} \cdot 1_{2g} \in \mathfrak{S}_g$ is just $U(g)$, we have

$$(23) \quad 2\pi\sqrt{-1}\bar{\lambda}_\sigma(Z_1, Z_2) = -\log_\sigma \det(Z_1 Z_2) + \log_\sigma \det(Z_1) + \log_\sigma \det(Z_2)$$

for $(Z_1, Z_2) \in U(g) \times U(g)$. By (23), the restriction of the cohomology class $[\bar{\lambda}_\sigma]$ to $U(g)$ is the pull-back of the cohomology class corresponding to the universal covering $0 \rightarrow \mathbb{Z} \rightarrow \tilde{U}(1) \cong \mathbb{R} \rightarrow U(1) \rightarrow 1$, via the map $\det : U(g) \rightarrow U(1)$. Since the induced map $(\det)_* : \pi_1(U(g)) \rightarrow \pi_1(U(1))$ is an isomorphism, we obtain $[\bar{\lambda}_\sigma] = u$. Since the cohomology class $[\bar{\lambda}_\sigma]$ is independent of the choice of the branch of $\log_\sigma j(\tau, \gamma)$ and since the restriction of u to Γ_g is the generator of the cohomology $H^2(\Gamma_g, \mathbb{Z})$ we obtain the assertion. \square

Lemma 7.5. *Let $g \geq 2$. The map $\delta : H^1(\pi_1(\mathfrak{S}'_g, *), \mathbb{Z}) \rightarrow H^2(\Gamma_g, \mathbb{Z})$ is given by*

$$(m, n) \mapsto (k_1(g)m + k_2(g)n) \in H^2(\Gamma_g, \mathbb{Z}) \cong \mathbb{Z}$$

for $(m, n) \in H^1(\pi_1(\mathfrak{S}'_g, *), \mathbb{Z}) \cong \mathbb{Z}^{\oplus 2}$. Here,

$$k_1(g) = 2^{g-2}(2^g + 1), \quad k_2(g) = \frac{(g+3) \cdot g!}{4} - 2^{g-3}(2^g + 1)$$

are the weights of Siegel modular forms $\chi_g(\tau), J_g(\tau)$, respectively.

Proof. Let $\sigma : \Gamma_g \rightarrow S_g$ be a section, and write $\sigma(\gamma) = [(l_\gamma, \gamma)] \in S_g$ for $\gamma \in \Gamma_g$. We can assume that $l_{\gamma^{-1}} = -\gamma \cdot l_\gamma$, where $-l(t) := l(1-t)$, $t \in [0, 1]$ for a path $l(t)$. Hence $\sigma(\gamma^{-1}) = \sigma(\gamma)^{-1}$. Let α be an element of $H^1(\pi_1(\mathfrak{S}'_g, *), \mathbb{Z})^{\Gamma_g} \cong \text{Hom}(\pi_1(\mathfrak{S}'_g, *), \mathbb{Z})^{\Gamma_g}$. Then $\delta(\alpha) : \Gamma_g \times \Gamma_g \rightarrow \mathbb{Z}$ is given by

$$(A, B) \mapsto \alpha(\sigma(A)\sigma(B)\sigma(AB)^{-1}) \in \mathbb{Z}, \quad (A, B) \in \Gamma_g \times \Gamma_g,$$

where we identify $\sigma(A)\sigma(B)\sigma(AB)^{-1} \in \text{Im}\{\pi_1(\mathfrak{S}'_g, *) \rightarrow S_g\}$ with the corresponding preimage of $\pi_1(\mathfrak{S}'_g, *)$ under the inclusion $\pi_1(\mathfrak{S}'_g, *) \hookrightarrow S_g$. Write $\sigma(A)\sigma(B)\sigma(AB)^{-1} = [(l_{(A,B)}, 1)] \in \pi_1(\mathfrak{S}'_g, *)$.

Here $l_{(A,B)}$ is a loop on \mathfrak{S}'_g , which is the composition of the paths l_B , $B^{-1} \cdot l_A$ and $-l_{AB}$. Under the identification $H^1(\pi_1(\mathfrak{S}'_g, *), \mathbb{Z})^{\Gamma_g} \cong \mathbb{Z}^{\oplus 2}$ given in Lemma 7.3, the cochain $\delta(m, n)$ is given by

$$\delta(m, n)(A, B) = \frac{1}{2\pi\sqrt{-1}} \int_{l_{(A,B)}} d\log\chi_g(\tau)^m J_g(\tau)^n \in \mathbb{Z}, \quad (A, B) \in \Gamma_g \times \Gamma_g,$$

for $(m, n) \in H^1(\pi_1(\mathfrak{S}'_g, *), \mathbb{Z})^{\Gamma_g} \cong \mathbb{Z}^{\oplus 2}$. Using σ , we choose the branch $\log_\sigma j(\tau, \gamma)$ for $\gamma \in \Gamma_g$ such that

$$\log_\sigma j(*, \gamma) := \frac{1}{k_1(g)} \int_{l_{\gamma^{-1}}} d\log\chi_g(\tau).$$

Then we get

$$\begin{aligned} 2\pi\sqrt{-1}\delta(1, 0)(A, B) &= \int_{l_{(A,B)}} d\log\chi_g(\tau) \\ &= \int_{AB \cdot l_{(A,B)}} d\log\chi_g(AB \cdot \tau) \\ &= \int_{AB \cdot l_{(A,B)}} [k_1(g)d\log j(\tau, AB) + d\log\chi_g(\tau)] \\ &= \int_{AB \cdot l_B} d\log\chi_g(\tau) + \int_{A \cdot l_A} d\log\chi_g(\tau) - \int_{AB \cdot l_{AB}} d\log\chi_g(\tau) \\ &= - \int_{l_{B^{-1}}} d\log\chi_g(A \cdot \tau) - \int_{l_{A^{-1}}} d\log\chi_g(\tau) + \int_{l_{(AB)^{-1}}} d\log\chi_g(\tau) \\ &= - \int_{l_{B^{-1}}} [k_1(g)d\log j(\tau, A) + d\log\chi_g(\tau)] \\ &\quad - k_1(g)\log_\sigma j(*, A) + k_1(g)\log_\sigma j(*, AB) \\ &= k_1(g)[- \log_\sigma j(B \cdot *, A) + \log_\sigma j(*, A) - \log_\sigma j(*, B) \\ &\quad - \log_\sigma j(*, A) + \log_\sigma j(*, AB)] \\ &= k_1(g)[\log_\sigma j(*, AB) - \log_\sigma j(B \cdot *, A) - \log_\sigma j(*, B)]. \end{aligned}$$

By Lemma 7.4 we get $\delta(1, 0) = k_1(g) \in H^2(\Gamma_g, \mathbb{Z}) \cong \mathbb{Z}$. Similarly, $\delta(0, 1) = k_2(g) \in H^2(\Gamma_g, \mathbb{Z}) \cong \mathbb{Z}$. This completes the proof. \square

Proof of Theorem 7.1. Since $H^1(\Gamma_g, \mathbb{Z})$ in the exact sequence (5), we get $H^1(S_g, \mathbb{Z}) = \ker \delta$. By Lemma 7.5, we get $\ker \delta = 0$ for $1 \leq g \leq 3$ and $\ker \delta \cong \mathbb{Z}$ for $g \geq 4$. This completes the proof of Theorem 7.1. \square

8. The value for the Dehn twist

In this section, we shall compute the value of Φ_{2g} for the *Dehn twist*, which is defined as follows (cf. [16]). Let $\Delta \subset \mathbb{C}$ be the unit disk. Recall that the Andreotti-Mayer locus \mathcal{N}_{2g} has two irreducible components $\theta_{null, 2g}$ and \mathcal{N}'_{2g} by Theorem 4.3. Let $\rho : \Delta \rightarrow \mathfrak{S}_{2g}$ be a C^∞ -map such that $\rho(0) \in \theta_{null, 2g}$ is a generic point, $\rho(z) \notin \mathcal{N}_{2g}$ for $z \in \Delta \setminus \{0\}$ and $\rho(\Delta)$ intersects with $\theta_{null, 2g}$ at $\rho(0)$ transversally. For simplicity we assume that the base point $*$ lies in $\rho(\partial\Delta)$ and we denote the monodromy corresponding to the loop $\rho|_{\partial\Delta} : \partial\Delta \rightarrow \mathfrak{S}'_{2g}$ by $\sigma_{2g} \in S_{2g}$. The element σ_{2g} is called the Dehn twist. We put

$$\varpi : X_{2g} := \Delta \times_\rho \Theta \rightarrow \Delta,$$

which is smooth family of theta divisors over Δ induced from the universal family $\pi : \Theta \rightarrow \mathfrak{S}_{2g}$ by ρ . Let $\tilde{\rho} : X_{2g} \rightarrow \Theta$ be the lift of the map ρ defined as the projection to the second factor. By

the assumption of ρ and the Theorem 4.3, $\text{Sing}(\varpi^{-1}(0))$ consists of one ordinary double point and $\varpi^{-1}(z)$ is a smooth theta divisor for $z \in \Delta \setminus \{0\}$. Notice that ∂X_{2g} endowed with the orientation induced from X_{2g} is diffeomorphic to the mapping torus $M_{\sigma_{2g}^{-1}}$ endowed with the natural orientation, i.e., $\partial X_{2g} = -M_{\sigma_{2g}}$.

Theorem 8.1. *The following equality holds:*

$$\Phi_{2g}(\sigma_{2g}) = \begin{cases} -\frac{4}{5} & \text{if } g = 1, \\ (-1)^{g+1} \frac{(2g+1)2^{2g+2}(2^{2g+2}-1)}{(2g+3)!} B_{g+1} & \text{if } g > 1. \end{cases}$$

Proof. Put $\Delta_r := \{z \in \Delta \mid |z| < r\} \subset \Delta$ for $0 < r < 1$. We choose ρ such that the restriction $\rho|_{\Delta_{1/3}} : \Delta_{1/3} \rightarrow \rho(\Delta_{1/3}) \subset \mathbb{S}_{2g}$ is a holomorphic embedding that

$$(24) \quad \rho(re^{\sqrt{-1}\theta}) = \rho\left(\frac{2}{3}e^{\sqrt{-1}\theta}\right), \quad \frac{2}{3} < r \leq 1, \quad 0 \leq \theta < 2\pi.$$

Let g^Δ be the metric on $T\Delta$ which is a product metric near the boundary $\partial\Delta$ and coincides with the metric $\rho^*g^{\mathbb{S}_g}$ on $\Delta_{1/3}$. Let $p \in X_{2g}$ be the unique singular point on the singular fiber X_0 . Let $g^{X_{2g}/\Delta}$ be the metric on $T(X_{2g}/\Delta)|_{X_{2g}-\{p\}}$ induced from the metric g^{Θ/\mathbb{S}_g} via the map ρ . Let $g^{X_{2g}}$ be the metric on TX_{2g} which coincides with $g^{X_{2g}/\Delta} \oplus \varpi^*g^\Delta$, where we used the connection induced from the connection $P_{\Theta'}$ on Θ' via the map ρ , on $X_{2g} - \{p\}$ and coincides with the metric induced from the metric $g^{\Theta'}$ via the map $\tilde{\rho}$ on a neighbourhood of p . Set

$$g^{X_{2g},\varepsilon} := g^{X_{2g}} \oplus \varepsilon^{-1} \varpi^* g^\Delta, \quad \varepsilon \in \mathbb{R}_{>0}.$$

By the assumption of g^Δ and the condition (24), $g^{X_{2g},\varepsilon}$ is the product metric near the boundary ∂X_{2g} for $\varepsilon \in \mathbb{R}_{>0}$. By the Atiyah-Patodi-Singer index theorem,

$$(25) \quad \text{Sign}(X_{2g}) = \int_{X_{2g}} L(TX_{2g}, g^{X_{2g},\varepsilon}) + \eta(M_{\sigma_{2g}}, g^{M_{\sigma_{2g}},\varepsilon}).$$

Here ∂X_{2g} is identified with $-M_{\sigma_{2g}}$, and $g^{M_{\sigma_{2g}},\varepsilon}$ is the restriction of $g^{X_{2g},\varepsilon}$ to the boundary $\partial X_{2g} \cong -M_{\sigma_{2g}}$. By the formula in [26], the first term of the right-hand side of (25):

$$(26) \quad \lim_{\varepsilon \rightarrow 0} L(TX_{2g}, g^{X_{2g},\varepsilon}) = L(T(X_{2g}/\Delta), \nabla^{X_{2g}/\Delta}) + P(-t, \dots, (-t)^{2g})|_{t^{2g}} \cdot \mu(p) \delta_p$$

Here $L(T(X_{2g}/\Delta), \nabla^{X_{2g}/\Delta})$ is only defined on $X_{2g} - \{p\}$ but has the natural smooth extension on whole X_{2g} . The constant $\mu(p)$ is the Milnor number of the singular point p , δ_p is the Dirac delta current supported at p and $P(x_1, \dots, x_{2g}) \in \mathbb{C}[[x_1, \dots, x_{2g}]]$ is defined by

$$\prod_{k=1}^{2g} L(x_k) = P(\sigma_1, \dots, \sigma_{2g}),$$

where $L(x) = x/\tanh(x)$ and $\sigma_1 = \sum_k x_k$, $\sigma_2 = \sum_{i>j} x_i x_j$, \dots , $\sigma_{2g} = \prod_k x_k$ are the fundamental symmetric polynomials. Notice that

$$P(-t, \dots, (-t)^{2g})|_{t^{2g}} = L^{-1}(t)|_{t^{2g}}.$$

Since p is a non-degenerate critical point of $\pi : X \rightarrow \Delta$, we get $\mu(p) = 1$, which together with (25), (26) and Theorem 4.7, yields that

$$(27) \quad \begin{aligned} \text{Sign}(X_{2g}) &= \frac{(-1)^g 2^{2g+1} (2^{2g+2} - 1)}{(g+1)(2g+1)} B_{g+1} \int_{\Delta} \rho^* dd^c \log \det \text{Im} \tau \\ &+ \frac{(-1)^g 2^{2g+2} (2^{2g+2} - 1)}{(2g+2)!} B_{g+1} + \eta^0(M_{\sigma_{2g}}, g^{M_{\sigma_{2g}},\varepsilon}). \end{aligned}$$

By (27) and Definition 6.1, we get

$$\begin{aligned}
\Phi_{2g}(\sigma_{2g}) &= \eta^0(M_{\sigma_{2g}}, g^{M_{\sigma_{2g}}, \epsilon}) \\
&+ \frac{(-1)^g 2^{2g+3} (2^{2g+2} - 1)}{(2g+3)!} B_{g+1} \int_{\partial \Delta} p^* d^c \left(\log |\Delta_{2g}(\tau)|^2 (\det \operatorname{Im} \tau)^{\frac{(2g+3) \cdot (2g)!}{2}} \right) \\
&= - \frac{(-1)^g 2^{2g+2} (2^{2g+2} - 1)}{(2g+2)!} B_{g+1} + \operatorname{Sign}(X_{2g}) \\
&+ \frac{(-1)^g 2^{2g+3} (2^{2g+2} - 1)}{(2g+3)!} B_{g+1} \int_{\Delta} p^* dd^c \log |\Delta_{2g}(\tau)|^2 \\
&= \frac{(-1)^{g+1} (2g+1) 2^{2g+2} (2^{2g+2} - 1)}{(2g+3)!} B_{g+1} + \operatorname{Sign}(X_{2g}),
\end{aligned}$$

where we used the Poincaré-Lelong formula and Theorem 4.4 to get the last equality. When $g = 1$, since the singular fiber has two irreducible components and $\operatorname{Sign}(X_2) = -1$, we obtain the proof for the case $g = 1$. We complete the proof by the following Lemma in the case $g > 1$. \square

Lemma 8.2. *Let $\pi : \mathfrak{X} \rightarrow \Delta$ be a Lefschetz degeneration of relative dimension $2n - 1$, i.e., π is a proper holomorphic surjective map from a $2n$ -dimensional complex manifold \mathfrak{X} to the unit disk Δ and there is a point $p \in \mathfrak{X}_0$ and an open neighbourhood $p \in U \cong \{(z_1, \dots, z_{2n}) \in \mathbb{C}^{2n} \mid \sum_{k=1}^{2n} |z_k|^2 < 1\}$ such that*

$$\pi(z_1, \dots, z_{2n}) = \sum_{k=1}^{2n} z_k^2, \quad (z_1, \dots, z_{2n}) \in U$$

and π_* has maximal rank on $\mathfrak{X} \setminus p$. Assume that $n > 1$. Then $\operatorname{Sign}(\mathfrak{X}) = 0$.

Proof. For $t \in \Delta$, we set $U_t := \mathfrak{X}_t \cap U$. Then a sequence of inclusions

$$\mathfrak{X}_0 \setminus U_0 \subset \mathfrak{X}_0 \setminus \{p\} \subset \mathfrak{X}_0 \subset \mathfrak{X}$$

induces a sequence of isomorphisms:

$$(28) \quad H_{2n}(\mathfrak{X}_0 \setminus U_0, \mathbb{Z}) \cong H_{2n}(\mathfrak{X}_0 \setminus \{p\}, \mathbb{Z}) \cong H_{2n}(\mathfrak{X}_0, \mathbb{Z}) \cong H_{2n}(\mathfrak{X}, \mathbb{Z}).$$

Here the first isomorphism follows from the homotopy equivalence of $\mathfrak{X}_0 \setminus U_0$ and $\mathfrak{X}_0 \setminus \{p\}$, the second isomorphism follows from the fact $\operatorname{codim}_{\mathbb{R}} \{p\} / \mathfrak{X}_0 = 4n - 2 > 2n + 1$, and the third isomorphism follows from the fact that the inclusion $\mathfrak{X}_0 \hookrightarrow \mathfrak{X}$ is a deformation retraction. By Ehresman's Theorem, $\mathfrak{X} \setminus U$ is diffeomorphic to $(\mathfrak{X}_0 \setminus U_0) \times \Delta$ as a fiber bundle over Δ . Since Δ is contractible, the inclusion $\mathfrak{X}_t \setminus U_t \hookrightarrow \mathfrak{X} \setminus U$ induces an isomorphism $H_{2n}(\mathfrak{X}_t \setminus U_t, \mathbb{Z}) \cong H_{2n}(\mathfrak{X} \setminus U, \mathbb{Z})$. By (28), the inclusion $\mathfrak{X}_t \setminus U_t \hookrightarrow \mathfrak{X}$ induces an isomorphism $H_{2n}(\mathfrak{X}_t \setminus U_t, \mathbb{Z}) \rightarrow H_{2n}(\mathfrak{X}, \mathbb{Z})$. Hence, for any $t \in \Delta$, any element of $H_{2n}(\mathfrak{X}, \mathbb{Z})$ can be represented by a cycle contained in \mathfrak{X}_t . Therefore the intersection matrix of $H_{2n}(\mathfrak{X}, \mathbb{Z})$ is trivial and $\operatorname{Sign}(\mathfrak{X}) = 0$. This completes the proof. \square

Remark 8.3. When $g = 1$, $\sigma_2 \in \mathcal{M}_2$ is the Dehn twist along a separating simple closed curve on a Riemann surface of genus two. Since $\operatorname{Sign}(X_2) = -1$ and $B_2 = \frac{1}{30}$, we obtain $\phi_2(\sigma_2) = \Phi_2(\sigma_2) = -\frac{4}{5}$, which confirms a result of Matsumoto ([19]).

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